

On Degenerated Monge-Ampere Equations over Closed Kähler Manifolds

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1 Introduction

The main goal for this note is to prove the following theorem which is an improved version of what is stated in [TZ].

Theorem 1.1. *Let X be closed a Kähler manifold with (complex) dimension $n \geq 2$. Suppose we have a holomorphic map $F : X \rightarrow \mathbb{CP}^N$ with the image $F(X)$ of the same dimension. Let ω_M ¹ be any Kähler form over some neighbourhood of $F(X)$ in \mathbb{CP}^N . For the following equation of Monge-Ampere type:*

$$(F^*\omega_M + \sqrt{-1}\partial\bar{\partial}u)^n = f\Omega, \quad (1.1)$$

where Ω is a fixed smooth (nondegenerated) volume form over X and f is a nonnegative function in $L^p(X)$ for some $p > 1$ with the correct total integral over X , i.e. $\int_X f\Omega = \int_X (F^*\omega_M)^n$,² we have the following:

- (1) (Apriori estimate) Suppose u is a weak solution in $PSH_{F^*\omega_M}(X) \cap L^\infty(X)$ of the equation with the normalization $\sup_X u = 0$, then there is a constant C such that $\|u\|_{L^\infty} \leq C\|f\|_{L^p}^n$ where C only depend on F , ω and p ;
- (2) There would always be a bounded solution for this equation;
- (3) If F is locally birational, then any bounded solution is actually the unique continuous solution.

The improvements are in two places:

- i) X closed Kähler instead of projective;
- ii) in statement (3) about continuity, the assumption is weakened a lot.

It might be worth taking a little time to clarify some terminologies appearing in the statement.

First, u is a weak solution means both sides are equal as (Borel) measure. The meaning of right hand side is classic with u being bounded.

¹The “ M ” is the initial letter of “*model*” since ω_M can be naturally understood as the model metric of original degenerated metric interested in and the degeneration information is hiding in the map F .

²This quantity is clearly positive from our assumption.

In definition of $L^p(X)$ space, we choose Ω as the volume form. The choice is clearly not so rigid.

In (3), “locally birational” means for a small enough neighbourhood U of any point on $F(X)$, each component of $F^{-1}(U)$ would be birational to U (under F). Clearly it would be the case if F is birational itself and in fact this is the case with most geometric interests as I see it now.

If F is an embedding, this theorem is proved in [K1] for even more general function f . Actually, he only needs the $F^*\omega_M$ to represent a Kähler class.

For proving the main theorem above, basically we generalize the original argument there which makes use of the results in [BT] and many other works in pluripotential theory.

Remark 1.2. *The discussion in this short note is supposed to be fairly concise. We achieve this by taking shortcuts which might make the idea less shown. The argument is complete except for frequently referring to Kolodziej’s [K1] and [K2]. For greater details and more related discussions, we refer the interested readers to [Z].*

In this note, plurisubharmonic sometimes means plurisubharmonic with respect to some background form. Hopefully, it’ll be clear from the context.

Acknowledgment 1.3. *For projective X , the main results of this paper except the continuity in (3) of Theorem 1.1 were announced and discussed in my previous preprint with Professor Tian. They were also presented in a talk by Tian at Imperial College on November 28, 2005. The general continuity was proved soon in January, 2006 after a few discussions with Professors S. Kolodziej and H. Rossi on approximating plurisubharmonic functions on singular spaces. I would like to thank them both for very useful discussions. Theorem 1.1 was also presented in my talk at Columbia University in February, 2006. A new result in the recent preprint by Blocki and Kolodziej allows the current generalization to a closed Kähler manifold X . I really appreciate their informing about this result. I would also like to thank my advisor, Professor Tian, for bringing up his attention to this basic question on complex Monge-Ampere equation.*

2 Idea of Generalizing Kolodziej’s Argument and Preparation

The (degenerated Monge-Ampere) equation we are considering is the following:

$$(\omega_\infty + \sqrt{-1}\partial\bar{\partial}u)^n = f\Omega$$

over X , where $\omega_\infty = F^*\omega_M$ ³, and Ω is a volume form over X . Our goal is to find a bounded (and even continuous) solution and get some properties for it,

³This ∞ illustrates the point of view that this degenerated case most naturally arises as the limit of nondegenerated case which has been used in [TZ].

for example, uniqueness.

2.1 Idea of Generalization

Basically, the solution is to be obtained by taking the limit of solutions for a family of approximation equations. Of course we want the solvability of the approximation equations to be known. In our case, the results in [K1] should be sufficient to guarantee this.

In order to get a limit, we need the apriori (L^∞) bound for those approximate solutions as in [K1]. In the original argument there, it is the generalized right hand side that is treated. We can deal with the right hand of the equation above just as well. The main difficulty now is of course the degeneracy of ω_∞ as a Kähler metric over X .

We have a natural family of approximation equations as follows:

$$(\omega_\infty + \epsilon\phi + \sqrt{-1}\partial\bar{\partial}u_\epsilon)^n = C_\epsilon f\Omega$$

where $\phi > 0$ be closed smooth $(1,1)$ -form and $\int_X C_\epsilon f\Omega = \int_X (\omega_\infty + \epsilon\phi)^n$ for $\epsilon \in (0, 1]$. Obviously, we have $C_\epsilon \in (1, C]$.

From Kolodziej's result, we have no trouble to find a unique continuous solution for each of these equations after requiring the normalization $\sup_X u_\epsilon = 0$. Now we just have to prove that u_ϵ 's are uniformly bounded from below.

The difficulty appears since $\omega_\infty + \epsilon\phi$ is not uniformly positive for $\epsilon \in (0, 1]$, i.e., if we consider the local potentials in coordinate balls, they will no longer be uniformly convex. Thus no matter how good the choice is, we do not have a uniform growth of the potential when moving away from the origin of the coordinate ball which is very crucial for the original argument.

This one blow seems to completely destroy Kolodziej's argument. As we see now, the main reason is that the picture of a coordinate (Euclidean) ball in X is a little too local.

The most important observation is that if one chooses a domain V which has those degenerated directions of ω_∞ going around inside, we can still have the uniform convex local potential, i.e., the values for the very outside part are greater than those of the very inside part by a uniform positive constant.

More precisely, if we take a ball in \mathbb{CP}^N which covers part of $F(X)$, then the preimage of that ball in X would be the domain V mentioned above. The potential of ω_∞ in V be convex in the sense above from the positivity of ω_M . Furthermore, we can see the domain V is hyperconvex in the usual sense which means we can have a continuous exhaustion of the domain, and there are actually a lot of nice plurisubharmonic functions over V which can be got by pulling back classic functions over the ball in \mathbb{CP}^N .

There seems to be another problem since the metric ϕ , which is used to perturb the equation, may (should) not have a global potential in the domain V . But we can deal with this by considering plurisubharmonic function in V

with respect to $\omega_\infty + \epsilon\phi$ for each $\epsilon \in (0, 1]$. In fact we can include the case of $\epsilon = 0$ in all the argument. The important thing is that our argument is uniform for all such ϵ 's.

Remark 2.1. *For the apriori estimate in (1) of the main theorem, we only need to work with ω_∞ ($\epsilon = 0$). But We'll need the estimate uniformly for all $\omega_\infty + \epsilon\phi$ for $\epsilon(0, 1]$ for the existence result in (2) of the main theorem. The above just says we can treat them together.*

We should point out that a global argument will be used below which might apparently hide the above idea of considering generalized domain V . Indeed the punchline is still to study the domain V . It should be quite natural that after getting all the necessary information for the local domains, we can patch them up to get for the whole of X just as in [K2].

2.2 Preparation

Many classic results in pluripotential theory are quite local, for example, weakly convergence results, and so can be used in our situation automatically. Many definitions also have their natural version for the domain V with background metric which can't be reduced to potential level globally in V , for example, relative capacity. Since it is of the most importance for us in this work, we give the definition below in the case when $V = X$, which is going to be used.

Definition 2.2. *Suppose ω is a (smooth) nonnegative $(1, 1)$ -form. For any (Borel) subset K of X , we define the relative capacity of K with respect to ϕ as follows:*

$$Cap_\omega(K) = \sup \left\{ \int_K (\phi + \sqrt{-1}\partial\bar{\partial}v)^n \mid v \in PSH_\phi(X), \quad -1 \leq v \leq 0 \right\}.$$

We require ω to be nonnegative so that $PSH_\omega(X)$ is not empty. We also point out that usually, it only takes to consider compact set K in order to study any set by approximation.

Remark 2.3. *Classic results in this business can be found in classic works as [L], [BT]. More recent works as [D], [K2] might be more convenient as reference.*

The only thing which is not so trivially adjusted to our situation might be comparison principle which is so important and has a global feature. There are several ways to deal with this situation. One of them seems to be the easiest to describe and in a sense it minimizes the modification of Kolodziej's argument for our case. So I will use it in this note. The other ways have their own interests and will be discussed in [Z].

Now let's state the version of comparison principle we are going to use later.

Proposition 2.4. *For X as above, suppose $u, v \in PSH_\omega(X) \cap L^\infty(X)$ where ω is a smooth nonnegative closed $(1, 1)$ -form, then*

$$\int_{\{v < u\}} (\omega + \sqrt{-1}\partial\bar{\partial}u)^n \leq \int_{\{v < u\}} (\omega + \sqrt{-1}\partial\bar{\partial}v)^n.$$

This version is slightly different from other more classic versions because X may not be projective, ω may not be positive and the functions may not be continuous. The description of justification is as follows.

Basically we still just need a decreasing approximation for any bounded plurisubharmonic function by smooth plurisubharmonic functions according to the argument in [BT]. This is not as easy as in Euclidean space where convolution is available. And the possible loss of projectivity of X makes it difficult to use some other classic results.

But according to the recent result of Blocki and Kolodziej in [BK], we can have a decreasing smooth approximation for plurisubharmonic function over X . The approximation result need the background form to be positive (i.e., a Kähler metric), but clearly nonnegative form (as ω_∞ for us) is acceptable when it comes down to comparison principle by simple approximation argument.⁴ This is also why we can now have X to be just Kähler instead of projective as stated in [TZ].

3 Apriori L^∞ Estimate

3.1 Bound Relative Capacity by Measure

In the following, ω is a (smooth) nonnegative closed $(1, 1)$ -form. Keep in mind that ω stands for $\omega_\infty + \epsilon\phi$ for any $\phi \in [0, 1]$. The constants do not depend on ϵ .

For $u, v \in PSH_\omega(X) \cap L^\infty(X)$ with $U(s) := \{u - s < v\} \neq \emptyset$ for $s \in [S, S + D]$. Also assume v is valued in $[0, C]$. Then $\forall w \in PSH_\omega(X)$ valued in $[-1, 0]$, for any $t \geq 0$, since $0 \leq t + Ct + tw - tv \leq t + Ct$, we have:

$$U(s) \subset V(s) = \{u - s - t - Ct < tw + (1 - t)v\} \subset U(s + t + Ct).$$

So we have for $0 < t \leq 1$:

$$\begin{aligned} \int_{U(s)} (\omega + \sqrt{-1}\partial\bar{\partial}w)^n &= t^{-n} \int_{U(s)} (t\omega + \sqrt{-1}\partial\bar{\partial}(tw))^n \\ &\leq t^{-n} \int_{U(s)} (t\omega + \sqrt{-1}\partial\bar{\partial}(tw) + (1-t)\omega + \sqrt{-1}\partial\bar{\partial}((1-t)v))^n \\ &= t^{-n} \int_{U(s)} (\omega + \sqrt{-1}\partial\bar{\partial}(tw + (1-t)v))^n \\ &\leq t^{-n} \int_{V(s)} (\omega + \sqrt{-1}\partial\bar{\partial}(tw + (1-t)v))^n \\ &\leq t^{-n} \int_{V(s)} (\omega + \sqrt{-1}\partial\bar{\partial}(u - s - t - Ct))^n \\ &\leq t^{-n} \int_{U(s+t+Ct)} (\omega + \sqrt{-1}\partial\bar{\partial}u)^n. \end{aligned} \tag{3.1}$$

⁴We do need X to be Kähler.

Comparison principle is applied to get the second to the last \leq . All the other steps are rather trivial from the setting.

Thus we conclude

$$Cap_\omega(U(s)) \leq t^n \int_{U(s+t+Ct)} (\omega + \sqrt{-1}\partial\bar{\partial}u)^n.$$

from the definition of Cap_ω .

Let's rewrite this inequality as:

$$Cap_\omega(U(s)) \leq (1+C)^n t^{-n} \int_{U(s+t)} (\omega + \sqrt{-1}\partial\bar{\partial}u)^n$$

for $t \in (0, \min(1, \frac{S+D-s}{1+C})]$. Of course, for our purpose, it is always safe to assume $\frac{S+D-s}{1+C} < 1$.

Intuitively, the constant D can be seen as the gap where the values of u can stretch over.

3.2 Bound Gap D by Capacity

We are still in the previous setting. Now assume that for any (Borel or compact) subset E of X , we have:

$$\int_E (\omega + \sqrt{-1}\partial\bar{\partial}u)^n \leq A \cdot \frac{Cap_\omega(E)}{Q(Cap_\omega(E)^{-\frac{1}{n}})}$$

for some constant $A > 0$, where $Q(r)$ is an increasing function for positive r with positive value. From now on, this condition will be denoted by Condition (A).

The result to be proved in this subsection is as follows:

$$D \leq \kappa(Cap_\omega(U(S+D)))$$

for the following function

$$\kappa(r) = C_n A^{\frac{1}{n}} \left(\int_{r^{-\frac{1}{n}}}^{\infty} y^{-1} (Q(y))^{-\frac{1}{n}} dy + (Q(r^{-\frac{1}{n}}))^{-\frac{1}{n}} \right),$$

where C_n is a positive constant only depending on n .

The proof is a little technical but quite elementary in spirit. We will briefly describe the idea below.

The previous part gives us an inequality as “ $Cap \leq measure$ ”.

Condition (A) gives the other direction “ $measure \leq Cap$ ”.

We can then combine them to get some information about the length of the interval which comes from t in the inequality proved before. The assumption of nonemptiness of the sets is needed because we have to divide $Cap_\omega(U(\cdot))$ from both sides in order to get something for t .

Finally, we can sum all these small t 's up to control for D .⁵

Of course we'd better use a delicate way to carry out all these just in sight of the rather complicated final expression of the function κ . It has been done beautifully in [K1].

Let's point out that in the argument, we do not have a positive lower bound for the t 's to be summed up, so it is important that the inequality proved in the previous part holds (uniformly) for all small enough $t > 0$.

3.3 Bound Capacity

For $u \in PSH_\omega(X) \cap L^\infty(X)$ and $u \leq 0$, suppose K is a compact set in X which can well be X itself, then there exists a positive constant C such that:

$$Cap_\omega(K \cap \{u < -j\}) \leq \frac{C\|u\|_{L^1(V)} + C}{j}.$$

Proof. For any $v \in PSH_\omega(X)$ and valued in $[-1, 0]$, consider any compact set $K' \subset K \cap \{u < -j\}$, using CLN inequality⁶ in [K2]:

$$\begin{aligned} \int_{K'} (\omega + \sqrt{-1}\partial\bar{\partial}v)^n &\leq \frac{1}{j} \int_K |u| (\omega + \sqrt{-1}\partial\bar{\partial}v)^n \\ &\leq \frac{C\|u\|_{L^1(V)} + C}{j}. \end{aligned}$$

From the definition of relative capacity, this would give the inequality above. \square

Now we consider the L^1 -norm for those approximation solutions u_ϵ (and also the solution u if it exists by assumption). The following is just the standard Green's function argument. Strictly speaking, the computation needs the function to be smooth, but we can achieve the final estimate by using approximation sequence given by the result in [BIK] for our situation. So suppose we have the regularity in the following.

For fixed $\epsilon \in [0, 1]$, suppose $u_\epsilon(x) = 0$ and $C > G$ where G is the Green function for the metric $\omega_1 = \omega_\infty + \phi$.

⁵We use the trivial fact that nonemptiness, nonzero (Lebesgue) measure and nonzero capacity are equivalent for such sets $U(s)$ from the fundamental properties of plurisubharmonic functions.

⁶The global version of this inequality over X is quite easy to justify in sight of the locality of the result.

Also since $\omega_\infty + \epsilon\phi + \sqrt{-1}\partial\bar{\partial}u_\epsilon \geq 0$, we have

$$\Delta_{\omega_1} u_\epsilon = \langle \omega_1, \sqrt{-1}\partial\bar{\partial}u_\epsilon \rangle \geq -\langle \omega_1, -\omega_\infty - \epsilon\phi \rangle \geq -C$$

where C is uniform for $\epsilon \in (0, 1]$. Basically, this tells that there should be no worry for the changing background metric.

Then we have:

$$\begin{aligned} 0 = u_\epsilon(x) &= \int_X u_\epsilon \omega_1^n + \int_{y \in X} G(x, y) \Delta_{\omega_1} u_\epsilon \cdot \omega_1^n \\ &= \int_X u_\epsilon \omega_1^n + \int_{y \in X} (G(x, y) - C) \Delta_{\omega_1} u_\epsilon \cdot \omega_1^n \\ &\leq \int_X u_\epsilon \omega_1^n - C \int_{y \in X} (G(x, y) - C) \omega_1^n \\ &\leq \int_X u_\epsilon \omega_1^n + C. \end{aligned} \tag{3.2}$$

This gives the uniform L^1 bound for u_ϵ 's by noticing they are all nonpositive.

Hence we know *the set where u_ϵ has very negative value should have (uniformly) small relative capacity.*

3.4 Conclusion

Combining all the results above, if we assume Condition (A) for some function $Q(r)$ and set the function v at the beginning to be 0, we have:

$$D \leq \kappa\left(\frac{C}{D}\right)$$

if $U(s) = \{u < -s\}$ nonempty for $s \in [-2D, -D]$ where C is a positive constant.

Furthermore, if we can choose the function $Q(r)$ to be $(1+r)^m$ for some $m > 0$ so that Condition (A) holds, this would imply that the function u only take values in a bounded interval since D can not be too large⁷.

That's enough for the lower bound in sight of the normalization $\sup_X u = 0$. The more explicit bound claimed in the theorem is not hard to get by carefully tracking down the relation. Of course the constant A in Condition (A) is fairly involved in this business.

3.5 Condition (A)

In this section, we justify Condition (A) under the measure assumption in the main theorem. This part is the essential generalization of Kolodziej's original argument.

⁷As D goes to ∞ , κ goes to 0.

In our case, $f \in L^p$ for some $p > 1$, which is the measure on the left hand side of Condition (A) from the equation we want to solve. For the approximation equations, the measures are different, but clearly we can bound the L^p -norm uniformly.

Applying Hölder inequality, we know that it suffices to prove the following inequality:

$$\lambda(K) \leq A \cdot (Cap_\omega(K)(1 + Cap_\omega(K)^{-\frac{1}{n}})^{-m})^q,$$

where λ is the smooth measure over X and q is some positive constant depending on p . Obviously, it would be enough to prove:

$$\lambda(K) \leq C_l \cdot Cap_\omega(K)^l \cdot \dots \cdot (A)$$

for l sufficiently large.

Of course we have $\lambda(K) < C$, so in fact we can get for any nonnegative l if the above is true. In the following, we'll consider Condition (A) in this form.

For ω (uniformly) positive, this can be easily reduce to a Euclidean ball. As in [K2], using a classic measure theoretic result in [Ts], we have:

$$\lambda(K) \leq C \cdot \exp\left(-\frac{C}{Cap_\omega(K)^{\frac{1}{n}}}\right).$$

This is actually stronger than the version above after noticing small capacity situation is of the main interest.

In the following proof of Condition (A), the essential step is to prove the following inequality:

$$\lambda(K) \leq C_1 \cdot \epsilon^{N_1} + C_1 \cdot \epsilon^{-N_2} \exp\left(\frac{C_2}{\log \epsilon \cdot Cap_\omega(K)^{\frac{1}{n}}}\right) \cdot \dots \cdot (B),$$

for sufficiently small $0 < \epsilon < 1$. All positive constants C_i 's do NOT depend on ϵ . This ϵ has nothing to do with the ϵ appearing before in $\omega_\infty + \epsilon\phi$.

After proving this, by putting $\epsilon = Cap_\omega(K)^\beta$ for properly chosen $\beta > 0$, we can justify Condition (A) for any chosen l by noticing the dominance of exponential growth over polynomial growth.

It is easy to notice that we can have uniform constants for all ω 's related once we get for ω_∞ from the favorable direction of the control we want. And we also only need to prove Condition (A) for sets close to the subvariety $\{\omega_\infty^n = 0\}$ in sight of the results in [K2].

The rest part of this section will be devoted to the proof of inequality (B). The following construction is of fundamental importance for this goal.

Let's start with a better description of the map $F : X \rightarrow F(X) \subset \mathbb{CP}^N$. For simplicity, we'll assume here that F provides a birational morphism between X and $F(X)$. This assumption will be removed at the end.

Using this assumption, we have subvarieties $Y \subset X$ and $Z \subset F(X)$ such that $X \setminus Y$ and $F(X) \setminus Z$ are isomorphic under F and $F(Y) = Z$. Clearly Z should contain the singular subvariety of $F(X)$. It's the situation near Y (or Z) that is of main interest to us.

Now we use finitely many unit coordinate balls on X to cover Y . The union of the half-unit balls will be called V . Then we take two finite sets of open subsets depending on $\epsilon > 0$ as follows:

$\{U_i\}, \{V_i\}$, with $i \in I$, finite coverings of $V \setminus W$, where W is the intersection of ϵ -neighbourhood of Y ⁸ with $F(X)$, such that each pair $V_i \subset U_i$ is in one of the chosen unit coordinate balls. Moreover, $F(U_i)$ and $F(V_i)$ are the intersections of $F(X)$ with balls of sizes $\frac{1}{2}\epsilon^C$ and $\frac{1}{6}\epsilon^C$ where $C > 0$ are chosen to be big enough to justify the above construction.

Clearly $|I|$ is controlled by $C \cdot \epsilon^{-N_2}$.

For any compact set K in V , we have the following computation:

$$\begin{aligned}
\lambda(K) &\leq \lambda(W) + \sum_{i \in I} \lambda(K \cap \bar{V}_i) \\
&\leq C \cdot \epsilon^{N_1} + \sum_{i \in I} C \cdot \exp\left(-\frac{C}{\text{Cap}(K \cap \bar{V}_i, U_i)^{\frac{1}{n}}}\right) \\
&\leq C \cdot \epsilon^{N_1} + \sum_{i \in I} C \cdot \exp\left(\frac{C}{\log \epsilon \cdot \text{Cap}_{\omega_\infty}(K \cap \bar{V}_i)^{\frac{1}{n}}}\right) \\
&\leq C \cdot \epsilon^{N_1} + \sum_{i \in I} C \cdot \exp\left(\frac{C}{\log \epsilon \cdot \text{Cap}_{\omega_\infty}(K)^{\frac{1}{n}}}\right) \\
&\leq C \cdot \epsilon^{N_1} + C \epsilon^{-N_2} \cdot \exp\left(\frac{C}{\log \epsilon \cdot \text{Cap}_{\omega_\infty}(K)^{\frac{1}{n}}}\right).
\end{aligned}$$

That's just what we want. C_1 and C_2 are used in the original statement of (B) since the C 's at different places have different affects on the magnitude of the final expression. Of course, the same C for each term in the big sum have to be really the same constant. In the following, we justify the computation above. The only nontrivial steps are the second and third ones.

The second one is the direct application of (\star) , the classic result in \mathbb{C}^n as V_i and U_i are in one of the finitely many unit coordinate balls which clearly can be taken as the unit Euclidean ball in a uniform way.

The third step uses the following inequality:

$$\text{Cap}(K \cap \bar{V}_i, U_i) \leq C \cdot (-\log \epsilon)^n \cdot \text{Cap}_{\omega_\infty}(K \cap \bar{V}_i).$$

⁸That's a neighbourhood of Y correspondent to the intersection of balls of radius ϵ in \mathbb{CP}^N covering Z .

This result also has its primitive version in classic pluripotential theory for domains in \mathbb{C}^n . Extension of plurisubharmonic function is all what we need to prove it as described below.

For any $v \in PSH(U_i)$ valued in $[-1, 0]$. If we can “extend” this function to an element $-C \log \epsilon \cdot \tilde{v}$ where \tilde{v} is plurisubharmonic with respect to ω_∞ valued in $[-1, 0]$ over X , and also make sure that the measures $(\sqrt{-1} \partial \bar{\partial} v)^n$ and $(\omega_\infty + \sqrt{-1} \partial \bar{\partial} \tilde{v})^n$ are the same over \tilde{V}_i , then this would clearly imply the inequality above from the definition of relative capacity.

The construction will be done mostly on $F(X)$. The function v can be considered over $F(U_i)$. We’ll “extend” it to a neighbourhood $F(X) \setminus O_i$ in \mathbb{CP}^N where O_i is a neighbourhood of \tilde{V}_i in U_i .

Let’s first extend it locally in \mathbb{CP}^N . We can safely assume that the construction happens in (finite) half-unit Euclidean balls in \mathbb{CP}^N which cover the variety Z and have ω_M defined on the correspondent unit balls. ω_M can be expressed in the level of potential and so the construction is merely about functions.

Consider the plurisubharmonic function

$$h = (\log(\frac{36|z|^2}{\epsilon^{2C}}))^+ - 2,$$

where the upper $+$ means taking maximum with 0, on the unit ball in \mathbb{CP}^N but with the coordinate system z centered at the center of $F(V_i)$. It’s easy to see the pullback of this function, still denoted by h , is plurisubharmonic and $\max(h, v)$ on U_i is equal to v near \tilde{V}_i and equal to h near ∂U_i . So this function extends v to the preimage of the unit ball in \mathbb{CP}^N while keeping the values near \tilde{V}_i .

Now we want to extend further to the whole of X . We still work on $F(X) \subset \mathbb{CP}^N$. And it only left to extend the function h for the remaining part where the value is less restrictive.

$|h|$ is bounded by $-C \cdot \log \epsilon$ in the unit ball. So we can have

$$\sqrt{-1} \partial \bar{\partial} h = -C \cdot \log \epsilon \cdot (\omega_M + \sqrt{-1} \partial \bar{\partial} H)$$

for H plurisubharmonic with respect to ω_M valued in $[-1, 0]$ in the unit ball. Then using the same argument as in [K2], we can extend H to (uniformly bounded) $\tilde{H} \in PSH_{\omega_M}(O)$, where O is a neighbourhood of $F(X)$, using the positivity of ω_M . Finally we just take $\tilde{v} = F^* \tilde{H}$.

This ends the argument for the case when $F : X \rightarrow F(X)$ is a birational map.

Now we want to remove the birationality condition. In fact, after removing proper subvarieties Y and $Z = F(Y)$ of X and $F(X)$ respectively, we can have $F : X \setminus Y \rightarrow F(X) \setminus Z$ is a finitely-sheeted covering map, since the map is clearly of full rank there and the finiteness of sheets can be seen by realizing the preimage of any point in $F(X \setminus Z)$ is a finite set of points.

Then it's easy to see the argument before would still work in this situation. Basically, we still have the construction before. Now the only difference is that now the numbers of small pieces U_i and V_i need to be multiplied by the number of sheets and this won't affect the argument too much.

Hence we get the apriori L^∞ bound in general.

4 Existence of Bounded Solution

Now we discuss the existence of a bounded solution. As suggested in Section 2, approximation of the original equation in the main theorem is used. The (uniform) apriori estimate got in the previous section would give us the uniform L^∞ bound for the approximation solutions u_ϵ there.

Thus exactly the same argument of taking limit as used in [K1] can be applied for our case to get a bounded solution for the original equation.

Remark 4.1. *Indeed, u_ϵ is essentially decreasing as $\epsilon \rightarrow 0$ which will make the limit easier to take. More details about this and some uniqueness results for bounded solutions will be provided in [Z].*

5 Continuity of Bounded Solution and Stability Result

In fact, we can prove that a bounded solution for the original equation is actually continuous with just a little more assumption.

The argument is almost in the same line as the argument in [K1] (using the L^∞ argument above).

We still just have to analyze the situation around a carefully chosen point. But now the point might be in the set $\{\omega_\infty^n = 0\}$. So in order to have convexity of the local potential of ω_∞ , we have to consider the domain in X which is the preimage of a ball in \mathbb{CP}^n under the map F . In other words, we do have to consider the domain V mentioned earlier.

For such a domain, where $\omega_\infty + \sqrt{-1}\partial\bar{\partial}u = \sqrt{-1}\partial\bar{\partial}U$ for a bounded plurisubharmonic function U , we don't have convolution which gives a decreasing smooth approximation of any plurisubharmonic function.

That's where we'll use the additional assumption of F being locally birational and the classic result in [FN]. The reason for requiring this local birationality would be clear below.

Basically, we want to be able to push forward the function U above to the singular domain in $F(X)$ in the most straightforward way (without averaging, etc.). The blowing-down picture would work. But it'll be OK if there are several components in X correspondent to the same singular domain in $F(X)$ as we only need to treat each of the components. In a sense, we just do not want any branching. Notice that the most interesting case of F being birational to its image falls right into this category.

Then it's quite straightforward to see the pushforward function F_*U is a weak plurisubharmonic function. One needs to prove that for any holomorphic map from unit disk in \mathbb{C} to $F(X)$, the pullback of F_*U is subharmonic. The idea is to reduce to map holomorphic with image in subvariety of $F(X)$ with decreasing dimension.

Now the classic result in [FN], pointed out to me by Professor Kolodziej, tells us that we can (locally) extend F_*U to a plurisubharmonic function in a ball of \mathbb{CP}^N which would be enough for us to go through local argument (for a properly chosen V). Basically now we can again use convolution to get the desired approximation.

Thus we can justify continuity for bounded solutions as in [K1]. The punch-line is as follows. Suppose $\{U_j\}$ is the sequence of smooth plurisubharmonic functions constructed above which are defined on a neighbourhood slightly larger than V which decreases to U pointwisely. Then by the construction in [K1], which is very local and can be easily adjusted to our case, we can prove the sets $\{U + c < U_j\}$ are nonempty and relatively compact⁹ inside V for all $c \in (0, a)$ for $a > 0$ and $j > j_0$.

The argument for L^∞ estimate before gives

$$\frac{a}{2} \leq \kappa(\text{Cap}(\{U + \frac{a}{2} < U_j\}, V)).$$

Remark 5.1. *Let's point out that we have to use local argument to get this inequality. For this, the justification of comparison principle would be different from the global case since there is now boundary to consider. Also the locality of the approximation above using the result in [FN] seems to make it difficult to get the approximations for two functions for the same domain.*

We can deal with this problem using another way mentioned in the discussion about comparison principle. Basically, we use a different definition of relative capacity where only continuous functions are considered in taking the supremum. Then together with the approximation we have for U , we can go through the local argument for the sets $\{U + c \leq U_j\}$. At this moment, it might be a little too distracting to carry out all the details which will appear in [Z].

We also notice that the relative capacity of the set $\{U + \frac{a}{2} < U_j\}$ would go to 0 as $j \rightarrow \infty$. This can be justified by the decreasing convergence and $\frac{a}{2} > 0$. At last, we can draw the contradiction by letting j goes to ∞ in the equality above because the right hand side is going to 0.

Remark 5.2. *The continuity can be achieved without the additional assumption if we can have a decreasing approximation of the solution by functions in $PSH_{\omega_\infty}(X) \cap C^\infty(X)$. There are all kinds of results in this direction, but somehow I feel semi-positivity of ω_∞ won't be sufficient.*

⁹For the relative compactness of the sets, strictly speaking, we have to use another function which is constructed from U linearly instead U itself in the definition as in [K1]. It's a little too tedious to describe the details here.

Finally, let's point out that the stability result for bounded (hence continuous) solutions in [K2] also hold in the current case. The original proof there works for us without any essential change.

We would like to mention that the stability argument there can almost be directly used to consider merely bounded solution. This is not pointless here since the continuity result needs a little bit more assumption than boundedness result as for now. But there is an inequality used there which bounds the measures of mixed terms from two plurisubharmonic functions by the measures from each of them. This inequality seems to be hard to prove for merely bounded functions. More discussions for this can also be found in [Z].

6 Application

The most useful application of the results above would be the L^∞ estimate. Combining it with other estimates from *PDE* methods (for example, maximum principle), we can further understand the solution.

Let's illustrate this by the application in Kähler-Ricci flow. This has been discussed in [TZ] which focuses on the maximum principle argument.

Consider the following Kähler-Ricci flow

$$\frac{\partial \tilde{\omega}_t}{\partial t} = -\text{Ric}(\tilde{\omega}_t) - \tilde{\omega}_t, \quad \tilde{\omega}_0 = \omega_0, \quad (6.1)$$

where ω_0 is any given Kähler metric and $\text{Ric}(\omega)$ denotes the Ricci form of ω , i.e., in the complex coordinates, $\text{Ric}(\omega) = \sqrt{-1}R_{i\bar{j}}dz^i \wedge d\bar{z}^j$ where $(R_{i\bar{j}})$ is the Ricci tensor of ω . Let $\omega_\infty = -\text{Ric}\Omega$ for a volume form Ω . Set $\omega_t = \omega_\infty + e^{-t}(\omega_0 - \omega_\infty)$ and $\tilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}u$, we can put (6.1) on the level of potential and more in the Monge-Ampere setting as:

$$(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n = e^{\frac{\partial u}{\partial t} + u}\Omega.$$

We have seen in [TZ] that the right-handed side has a uniform L^p -bound for all t with any $1 < p \leq \infty$.

Now suppose $[\omega_\infty] = -K_X$ is nef and big. Hence it would be semi-ample and fall right into the setting of the main theorem.

Though generally in $\omega_t = \omega_\infty + e^{-t}(\omega_0 - \omega_\infty)$, it may not be true that $\omega_0 - \omega_\infty > 0$. Combining with the degenerated lower bound of u , we can still have uniform L^∞ estimate for $u(t, \cdot)$ with $t \in [0, \infty)$ simply by using part of ω_∞ in the front to dominate the second term.¹⁰ This would give us the boundedness of the limit and the continuity follows as well since we can choose the map F to be birational to its image for this case.

There would be generalization and more application in [Z].

¹⁰In fact, the uniqueness result in [TZ] allows us to only consider the case when $\omega_0 > \omega_\infty$.

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